ON OSCILLATORLIKE HAMILTONIANS AND SQUEEZING[1]

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Abstract

Generalizing a recent proposal leading to one-parameter families of Hamiltonians and to new sets of *squeezed* states, we construct larger classes of physically admissible Hamiltonians permitting new developments in squeezing. Coherence is also discussed.

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1 Introduction

We have recently proposed new sets of Fock states [1] which can be exploited in the contexts of coherence [2] and squeezing [3]-[5]. Due to the inclusion of a (real continuous) parameter λ (in the bosonic creation Heisenberg operator), the corresponding oscillatorlike "Hamiltonians" led to stationary Schrödinger problems characterized by λ-independent eigenvalues but λ-*dependent eigenfunctions*, the latter ones being particularly interesting [1] for the study of new *squeezed* states [3]. Moreover, this approach was in a certain sense a kind of deformation of the current one but following Wigner's point of view [6].

Let us insist strongly on the fact that we were considering [1] "squeezing" through the λ-dependent eigenfunctions of our Schrödinger problems and, evidently, through the associated meanvalues of position and momentum, their variances and (in)equalites coming from the Heisenberg relations. This differs from Yuen's approach [3] which is based on the study of "squeezing" through the famous two-photon coherent states of the radiation field asking for eigenstates of the oscillator operators and not of the Hamiltonian.

Here we want to generalize such developments by including a priori more than one parameter when, simultaneously, we study new creation *and* annihilation operators as well as the corresponding oscillatorlike "Hamiltonians" appearing as physically admissible or nonadmissible ones.

The contents are then distributed as follows. In *Section 2*, we recall a few relations issued from our first approach [1]. *Section 3* is devoted to its generalization already suggested. In *Section 4*, we apply these considerations to the squeezing problem and find real improvements with respect to the one-parameter previous results. Finally some conclusions and comments are included in *Section 5*.

2 A short survey of our recent proposal

Let us define the new (bosonic) creation Heisenberg operator by

$$a_\lambda^\dagger \equiv a^\dagger + \lambda I \quad , \lambda \in R, \tag{1}$$

where λ refers to a real continuous parameter and where $a^\dagger$ is the Hermitian conjugate of the annihilation operator a satisfying altogether the expected Heisenberg commutation relations, i.e.

$$[a, a_\lambda^\dagger] = I, \ [a, a] = [a_\lambda^\dagger, a_\lambda^\dagger] = 0. \tag{2}$$

These quantum harmonic oscillatorlike considerations lead to an analog of a (non-Hermitian) Hamiltonian of the type

$$H_\lambda = \frac{1}{2}\{a, a_\lambda^\dagger\} = \frac{1}{2}\{a, a^\dagger\} + \lambda a = H_{H.O.} + \lambda a \tag{3}$$

where the harmonic oscillator Hamiltonian is obviously given by

$$H_{H.O.} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 \ , \ H_{H.O.}^\dagger = H_{H.O.}. \tag{4}$$

Moreover they ensure that

$$[H_\lambda, a] = -a, \ [H_\lambda, a_\lambda^\dagger] = a_\lambda^\dagger, \tag{5}$$

so that (generalized) Wigner's approach [6] of quantum mechanics is still working. The energy eigenvalues and eigenfunctions have been determined as

$$E_{n,\lambda} = n + \frac{1}{2} \ \ (n = 0, 1, 2, ...) \tag{6}$$

and

$$\psi_{n,\lambda} = \frac{2^{-\frac{n}{2}}\pi^{-\frac{1}{4}}}{\sqrt{n!}\sqrt{L_n^{(0)}(-\lambda^2)}} e^{-\frac{x^2}{2}} H_n(x + \frac{\lambda}{\sqrt{2}}) \tag{7}$$

where, as usual, we have chosen units such that $\omega = 1$, $\bar{h} = 1$ and where H_n and $L_n^{(0)}$ refer to Hermite and generalized Laguerre polynomials [7], respectively. Let us insist on the *unchanged* spectrum (6) with respect to well known oscillator results but now with λ-modified eigenfunctions. Moreover we have shown [1] that these new eigenfunctions (7) correspond to specific *squeezed* states ([3],[4]). Let us recall that squeezed states have already been experimentally detected [5] being seen as "two-photon coherent states" for the electromagnetic field. Our new states lead to the characteristic inequality for *squeezing* given by

$$(\Delta x)_\lambda^2 = 2n + \frac{1}{2} - (2\lambda^2 + 1)\frac{L_{n-1}^{(1)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)} - 2\lambda^2(\frac{L_n^{(1)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)})^2 < \frac{1}{2} \tag{8}$$

if

$$n = 1, 2, 3, ... \; and \; \lambda \in R \setminus \,] - r, +r[\; , \; r \to 0 \; if \; n \to \infty. \tag{9}$$

3 A simple way to get generalized developments

The qualities and defects of our above approach [1] suggest the following new position of the problem:

to search for (bosonic) *oscillatorlike* annihilation (b) and creation (b^+) operators ensuring the following conditions

$$[b, b^+] = 1 \; , \; [H, b] = -b \; , \; [H, b^+] = b^+ \tag{10}$$

and

$$H = \frac{1}{2}\{b, b^+\} = \alpha \frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(x), \tag{11}$$

where b and b^+ have to be general expressions of the usual operators a and $a^\dagger$. Let us note that we have introduced different notations for the usual Hermitian conjugate operator $a^\dagger$ of a and the so-called b^+ associated to b with a general meaning discussed in the following.

Such a set of conditions obviously *contains* the Heisenberg and Wigner requirements through eqs. (10) and, moreover, restricts the Hamiltonian to Schrödingerlike ones through eq. (11) where α is a real constant and β, γ are arbitrary real functions of the space variable.

According to such a programme, let us introduce the generalized operators b and b^+ in terms of a and $a^\dagger$ by the following definitions

$$b = (1 + c_1)a + c_2 a^\dagger + c_3 \tag{12}$$

and

$$b^+ = c_4 a + (1 + c_5)a^\dagger + c_6, \tag{13}$$

where $c_1, c_2, ..., c_6$ are arbitrary (real) parameters and where the current harmonic oscillator context has been included by equating all the parameters to zero while the definition (1) is also incorporated in equating only c_6 with the λ-parameter, all the other c's being identically zero.

At this stage let us point out that the generalized operators (12) and (13) are intimately connected with the construction of the so-called "two-photon coherent states" due to Yuen [3] but here from inhomogeneous linear canonical transformations which could be summarized by the following form

$$\begin{pmatrix} b \\ b^{+} \end{pmatrix} = \begin{pmatrix} 1+c_1 & c_2 \\ c_4 & 1+c_5 \end{pmatrix} \begin{pmatrix} a \\ a^{\dagger} \end{pmatrix} + \begin{pmatrix} c_3 \\ c_6 \end{pmatrix}$$

where the matrix has to be invertible so that, for example,

$$(1+c_1)(1+c_5) - c_2 c_4 = 1.$$

This leads to

$$c_1 + c_5 + c_1 c_5 - c_2 c_4 = 0 \tag{14}$$

which is the constraint between the c's issued from (10) and leaving, in fact, only five independent parameters in the whole discussion.

By taking care of the definitions (12) and (13) in the Hamiltonian (11) and by remembering that

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right) \ , \ a^{\dagger} = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right), \tag{15}$$

the possible Hamiltonians are then found on the following form

$$H = A\frac{d^2}{dx^2} + (Bx + C)\frac{d}{dx} + Dx^2 + Ex + F \tag{16}$$

transferring the parametrization on the six parameters $A, B, ..., F$ given by

$$\begin{aligned}
A &= -\frac{1}{2} - c_2 c_4 + \frac{1}{2}c_4(1+c_1) + \frac{1}{2}c_2(1+c_5), \\
B &= c_4(1+c_1) - c_2(1+c_5), \\
C &= \frac{1}{\sqrt{2}}[c_6(c_1 - c_2 + 1) + c_3(c_4 - c_5 - 1)], \\
D &= \frac{1}{2} + c_2 c_4 + \frac{1}{2}c_4(1+c_1) + \frac{1}{2}c_2(1+c_5), \\
E &= \frac{1}{\sqrt{2}}[c_6(c_1 + c_2 + 1) + c_3(c_4 + c_5 + 1)], \\
F &= \frac{1}{2}c_4(1+c_1) - \frac{1}{2}c_2(1+c_5) + c_3 c_6.
\end{aligned} \tag{17}$$

These developments compared to the previous ones [1] clearly appear as a generalization; moreover it permits an interesting discussion at the level of physically admissible Hamiltonians as well as at the level of coherence and (or) squeezing, once we have solved the eigenvalue and eigenfunction problems associated with such Hamiltonians.

In terms of the new parameters, let us point out again that the current harmonic oscillator Hamiltonian (4) corresponds to

$$A = -D = -\frac{1}{2}\ ,\ B = C = E = F = 0, \tag{18}$$

while our previous deformation (1) leading to the Hamiltonian (3) is given by

$$A = -D = -\frac{1}{2}\ ,\ B = F = 0\ ,\ C = E = \frac{\lambda}{\sqrt{2}}. \tag{19}$$

With the Hamiltonian (16) and the relations (17), the stationary Schrödinger problem can now be solved by conventional quantum mechanical methods [8]. It leads to the general answer

$$E_n = F - \frac{B}{2} - \frac{C^2}{4A} - \frac{A}{p^2}(2n+1) + q^2(D - \frac{B^2}{4A}) + q(E - \frac{BC}{2A}) \tag{20}$$

while the corresponding eigenfunctions take the form

$$\psi_n(x) = exp[-\frac{B}{4A}x^2 - \frac{C}{2A}x]exp[-\frac{x^2}{2p^2} + \frac{qx}{p^2} - \frac{q^2}{2p^2}]H_n(\frac{x-q}{p}), \tag{21}$$

where p and q enter the necessary change of variable

$$x = py + q,\ (p \neq 0). \tag{22}$$

Let us mention the two constraints

$$\frac{p^4}{A}(D - \frac{B^2}{4A}) = -1 \tag{23}$$

and

$$2q(D - \frac{B^2}{4A}) + E - \frac{BC}{2A} = 0, \tag{24}$$

issued from these calculations.Together with eqs. (17), these relations (23), (24) fix the parameters p and q of our change of variable (22) to the unique values:

$$p^2 = -2A \ , \ q = 2EA - BC \tag{25}$$

in order to get in particular a positive spectrum. By requiring to deal with square integrable eigenfunctions, we finally have to ask for

$$A < 0 \quad and \quad B < 1 \tag{26}$$

compatible with the specific cases (18) and (19). We thus get (up to a normalization factor N_n) the solutions (21) as given by

$$\psi_n(x) = N_n exp[\frac{1-B}{4A}x^2]exp[(\frac{(B-1)C}{2A} - E)x]exp[\frac{(2EA-BC)^2}{4A}]$$
$$H_n(\frac{1}{\sqrt{-2A}}(x - 2EA + BC)). \tag{27}$$

Moreover we obtain the remarkable result of an *unchanged* real spectrum

$$E_n = n + \frac{1}{2} \ , \ n = 0, 1, 2, ..., \tag{28}$$

inside this general context as it was already the case in our first study (see (6)). Let us here insist on this real character without having required the selfadjointness of the Hamiltonian (16), a similar property to the one which has recently been quoted by Bender and Boettcher [9] although we have not required any specific discrete symmetries. Nevertheless, we have also noticed that a *necessary* and *sufficient* condition ensuring that $H^\dagger = H$ is simply

$$B = C = 0 \tag{29}$$

leading to a large class of *physically* admissible Hamiltonians.

As a last remark in this Section, let us point out that such eigenfunctions $\psi_n(x)$ like (27) are once again associated with Fock states - let us call them $\mid n >_c$ referring to the c-parametrization included in eqs.(12) and (13) - and it is interesting to quote the action of b and $b^\dagger$ on such states. We obtain

$$b \mid n >_c = \frac{n}{\sqrt{-A}} \frac{N_n}{N_{n-1}} (1 + c_1 - c_2) \mid n - 1 >_c \tag{30}$$

and

$$b^+ \mid n >_c = \frac{1}{2\sqrt{-A}} \frac{N_n}{N_{n+1}} (1 + c_5 - c_4) \mid n+1 >_c \tag{31}$$

and point out that

$$bb^+ \mid n >_c = (n+1) \mid n >_c \ , \ b^+ b \mid n >_c = n \mid n >_c \tag{32}$$

so that the conditions (10) and (11) are obviously satisfied, ensuring in particular that

$$\{b, b^+\} \mid n >_c = 2H \mid n >_c = (2n+1) \mid n >_c \ . \tag{33}$$

4 On implications in squeezing

Let us, *first*, extract new information by considering the lowest energy eigenvalue E_0 of the spectrum and its associated eigenfunction $\psi_0(x)$. Then, as a *second* step, let us come back very briefly on known cases corresponding to the Hamiltonian (16) with the conditions (18) or (19). Finally, let us consider *thirdly* new parametrizations exploiting the results obtained in Section 3 mainly with a view of interesting improvements in squeezing.

4.1 From the lowest eigenvalue of the spectrum

Due to the fundamental and specific role played by the lowest energy eigenvalue E_0, let us study coherence and squeezing through the eigenfunction $\psi_0(x) \equiv (27)$ which takes the explicit form

$$\psi_0(x) = N_0 exp[-\frac{1}{2}\alpha x^2 - \beta x - \frac{1}{2}\gamma] \ , \ N_0 = (\frac{\alpha}{\pi})^{\frac{1}{4}} exp[\frac{\alpha\gamma - \beta^2}{2\alpha}] \tag{34}$$

in order to ensure that

$$\int_{-\infty}^{+\infty} \psi_0^2(x) dx = 1 \tag{35}$$

where, for brevity, we have introduced the notations

$$\alpha = \frac{B-1}{2A} \ , \ \beta = E - \frac{(B-1)C}{2A} \ , \ \gamma = -\frac{1}{2A}(2EA - BC)^2. \tag{36}$$

In such a $n=0$-context, the meanvalues and consequences are readily obtained as follows:

$$\langle x\rangle_0 = \frac{2EA+C}{1-B}\ ,\ \langle x^2\rangle_0 = \frac{A}{B-1} + (\frac{2EA}{B-1} - C)^2,$$
$$\langle p\rangle_0 = 0\ ,\ \langle p^2\rangle_0 = \frac{B-1}{4A}\ , \tag{37}$$

so that we get

$$(\Delta x)^2_0 = \frac{A}{B-1}\ ,\ (\Delta p)^2_0 = \frac{B-1}{4A}. \tag{38}$$

These results ensure *coherence* due to the Heisenberg relation

$$(\Delta x)_0(\Delta p)_0 = \frac{1}{2} \tag{39}$$

and *squeezing* on the x-variable

$$(\Delta x)^2_0 < \frac{1}{2} \quad iff \quad B < 2A+1 \tag{40}$$

or on the p-variable

$$(\Delta p)^2_0 < \frac{1}{2} \quad iff \quad B > 2A+1. \tag{41}$$

Such inequalities on A and B only will suggest our future parametrizations in Sections 4.2 and 4.3. In fact, let us immediately inform the reader that we plan to priviledge the discussion on the x-variable so that eq. (40) will play the main role.

4.2 From known cases

(i) The *harmonic oscillator* context characterized by the condition (18) is obviously well known as far as coherence and squeezing are visited ([2]-[5]). As already mentioned, this case is contained in our study but we learn only that it corresponds to all c's equal to zero in eqs. (12) and (13), it generates a selfadjoint Hamiltonian (4) and deals with hermitian conjugated operators $b \equiv a$ and $b^\dagger \equiv a^\dagger$ verifying the condition

$$(b^\dagger)^\dagger = b. \tag{42}$$

(ii) The *deformed* context characterized by the condition (19) has already been discussed in [1]: it breaks down the condition (42) *and* the selfadjointness of the Hamiltonian (3) so that physical connections are here questionable although they correspond to *real* spectra and to new possibilities of squeezing for $n \neq 0$ [1]. Let us point out that the conditions (29) are obviously in contradiction with eqs. (19) and that, for $n = 0$, the inequalities (40) cannot be satisfied.

4.3 To new contexts

By keeping the conditions (29) in order to maintain the selfadjointness of the Hamiltonians, we can also require the condition (42). The latter leads to very simple demands of the types:

$$c_1 = c_5 \ , \ c_2 = c_4 \ , \ c_3 = c_6 \tag{43}$$

so that we then get families of *physically admissible Hamiltonians* which can be further exploited.

(i) Within such conditions, let us go to a one-parameter λ-deformation with, for example, the values

$$c_1 = c_5 = \frac{2}{3} \ , \ c_2 = c_4 = \frac{4}{3} \ , \ c_3 = c_6 = \lambda. \tag{44}$$

Such a case corresponds to the parametrization (17) given by

$$A = -\frac{1}{18} \ , \ C = \frac{9}{2} \ , \ E = 3\sqrt{2}\lambda \ , \ F = \lambda^2 \ , \ B = C = 0 \tag{45}$$

and the eigenvalues and eigenfunctions problem can be completely solved. We evidently get the spectrum (28) and the eigenfunctions (27) take the final form

$$\psi_n(x) = N_n exp[-\frac{9}{2}x^2 - \frac{6}{\sqrt{2}}\lambda x - \lambda^2] \ H_n(3x + \sqrt{2}\lambda) \tag{46}$$

where the normalization factor is found on the following form

$$N_n = \frac{\sqrt{3}\pi^{-\frac{1}{4}}2^{-\frac{n}{2}}}{\sqrt{n!}}. \tag{47}$$

Meanvalues and Heisenberg constraints can then be evaluated and we get

$$\langle x\rangle_\lambda = -\frac{\sqrt{2}}{3}\lambda \ , \ \langle x^2\rangle_\lambda = \frac{1}{9}(2\lambda^2 + n + \frac{1}{2}) \tag{48}$$

and

$$\langle p\rangle_\lambda = 0 \ , \ \langle p^2\rangle_\lambda = 9(n + \frac{1}{2}), \tag{49}$$

so that

$$(\Delta x)^2_\lambda = \frac{1}{9}(n + \frac{1}{2}) \ , \ (\Delta p)^2_\lambda = 9(n + \frac{1}{2}) \tag{50}$$

leading to

$$(\Delta x)_\lambda (\Delta p)_\lambda = n + \frac{1}{2}. \tag{51}$$

This result is analogous to the one of the *undeformed* case but, here, it permits, moreover, squeezing (but on x *only*) due to the relations (40) and (50). In fact, such a squeezing can only take place for $n = 0, 1, 2, 3$.

(ii) A final improvement of this example consists in the possible increase of such n-values permitting the squeezing and maintaining the nice property (42) and the selfadjointness of H. This can be realized through the new λ-deformation ($\lambda > 0$) characterized by the values

$$c_1 = c_5 = \frac{(\sqrt{\lambda} - 1)^2}{2\sqrt{\lambda}} \ , \ c_2 = c_4 = \frac{\lambda - 1}{2\sqrt{\lambda}} \ , \ c_3 = c_6 = 0, \tag{52}$$

leading to the relations (17) given now on the form

$$A = -\frac{1}{2\lambda} \ , \ D = \frac{\lambda}{2} \ , \ B = C = E = F = 0. \tag{53}$$

satisfying once again the inequalities (40) when $n = 0$.

Here the eigenfunctions are found as

$$\psi_n(x) = N_n exp[-\frac{\lambda}{2}x^2] \ H_n(\sqrt{\lambda}x) \ , \ N_n = \frac{\lambda^{\frac{1}{4}}\pi^{-\frac{1}{4}}2^{\frac{n}{2}}}{\sqrt{n!}} \tag{54}$$

and we get in correspondence with eqs. (50)

$$(\Delta x)^2_\lambda = \frac{1}{\lambda}(n + \frac{1}{2}) \ , \ (\Delta p)^2_\lambda = \lambda(n + \frac{1}{2}). \tag{55}$$

We thus notice once more the validity of eq. (51) ensuring *coherence* for the particular value $n = 0$ only but *squeezing* (in the x-coordinate) for all the values n satisfying the following inequality

$$\lambda > 2n + 1 > 0. \tag{56}$$

A further interesting property of the above eigenfunctions (54) (and evidently (46)) is that, due to the characteristics of Hermite polynomials [7], these solutions are not only normalized but are also *orthogonal* as it can be easily established.

If physical applications require a fixed *finite* set of levels in the energy spectrum, we can always choose, due to the inequality (56), our λ-parameter in order to guarantee the squeezing up to this n-value.

(iii) As a last context, let us relax the condition (42) and the selfadjointness of the Hamiltonian. This corresponds to an extension of the context discussed in [1] and recalled here in the subsection (4.2.ii). We can choose, for example,

$$b = a + \lambda a^\dagger \ , \ b^+ = a^\dagger \tag{57}$$

corresponding to all the null parameters c except $c_2 = \lambda$ or to

$$A = \frac{1}{2}(\lambda - 1) \ , \ B = -\lambda \ , \ C = E = 0 \ , \ D = \frac{1}{2}(\lambda + 1) \ , \ F = -\frac{\lambda}{2} \tag{58}$$

ensuring squeezing on x in the $n = 0$-case if $1 > \lambda > 0$. With the spectrum (28), the associated eigenfunctions here take the form

$$\psi_n(x) = N_n exp[-\frac{1}{2}(\frac{1+\lambda}{1-\lambda})x^2] \ H_n(\frac{x}{\sqrt{1-\lambda}}). \tag{59}$$

They are normalizable with

$$N_n = \frac{\pi^{-\frac{1}{4}}}{n!}(\frac{1+\lambda}{1-\lambda})^{\frac{1}{4}}(1+\lambda)^{\frac{n}{2}} F_n^{-\frac{1}{2}}(\lambda) \tag{60}$$

but not orthogonal. Depending on the *even* or *odd* character of n the functions $F_n(\lambda)$ are respectively given by

$$F_n(\lambda) = \sum_{l=0}^{\frac{n}{2}} \frac{2^{2l}\lambda^{n-2l}}{(2l)![(\frac{n}{2}-l)!]^2} \ \ , \ (n \ \ even), \tag{61}$$

or

$$F_n(\lambda) = \sum_{l=0}^{\frac{n-1}{2}} \frac{2^{2l+1}\lambda^{n-1-2l}}{(2l+1)![(\frac{n-1}{2}-l)!]^2} \ , \ (n \ \ odd). \tag{62}$$

These functions enter the evaluation of meanvalues and Heisenberg constraints for each n-value. Specific values are of interest in order to learn the general behaviour of the corresponding meanvalues and their consequences but these are only exercises. Let us just point out that, for $n = 0$, we get

$$\begin{aligned} \langle x\rangle_\lambda = 0 \ , \ \langle x^2\rangle_\lambda = \frac{1}{2}(\frac{1-\lambda}{1+\lambda}) = (\Delta x)^2_\lambda \\ \langle p\rangle_\lambda = 0 \ , \ \langle p^2\rangle_\lambda = \frac{1}{2}(\frac{1+\lambda}{1-\lambda}) = (\Delta p)^2_\lambda \end{aligned} \tag{63}$$

giving us *coherence* due to

$$(\Delta x)^2_\lambda (\Delta p)^2_\lambda = \frac{1}{4} \quad , \quad \forall\lambda, \tag{64}$$

while *squeezing* requires parametrizations according to

$$1 > \lambda > 0 \quad or \quad -1 < \lambda < 0 \tag{65}$$

in the x- *or* p- context respectively. Coherence is then lost if $n \neq 0$ but squeezing can be installed when specific refined inequalities of the type (65) are valid. The upper and lower bounds on these λ-values can be determined by entering the results (59)-(62) depending on the n-values we are considering.

5 Some further conclusions and comments

Among the above results, let us point out those obtained more particularly in the subsection (4.3.ii) leading to an attracting class of one-parameter selfadjoint Hamiltonians

$$H_\lambda = \frac{1}{2}\{b_\lambda, b^\dagger_\lambda\}, \tag{66}$$

with

$$b_\lambda = (1 + \frac{(\sqrt{\lambda}-1)^2}{2\sqrt{\lambda}})a + \frac{\lambda-1}{2\sqrt{\lambda}}a^\dagger \ , \ b^\dagger_\lambda = b^+ \ , \ (b^\dagger_\lambda)^\dagger = b_\lambda, \tag{67}$$

characterized by a deformation parameter $\lambda > 0$ and corresponding to the current harmonic oscillator case when $\lambda = 1$. Appearing nearly as a trivial result, this family opens possible new studies of squeezing through energy eigenfunctions (54) which are not only normalizable but also orthogonal among themselves. The possible choice $\lambda > 2n + 1$ with a fixed set of energy eigenvalues given by the usual spectrum (28) is maybe an interesting connection with possible experimental realizations for oscillatorlike systems in order to test and to realize the associated squeezed states.

One further comment is the possible exploitation of our generalized operators b and b^+ by studying more than one (real) parameter in the definitions (12) and (13) as just noticed elsewhere [11].

Another point which has to be recalled is that the motivations of deforming our (annihilation and creation) oscillatorlike operators (as realized in eqs. (12) and (13)) were intimately connected with a specific mathematical property called "subnormality of operators" [10], a property already exploited in our previous letter [1].

As a final comment, let us also recall that our developments could evidently be extended to the fermionic sector as already specified in our first approach [1]. The generalizations (12) and (13) can be realized on *fermionic* annihilation and creation operators and their consequences can then be deduced. Then the superposition of these bosonic and fermionic contexts could be considered in order to go towards supersymmetric developments [12] by including simultaneously specific physical as well as mathematical properties.

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